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Noncommutative geometry, quantum effects and DBI-scaling in the collapse of D0-D2 bound states

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Abstract: We study fluctuations of time-dependent fuzzy two-sphere solutions of the non-abelian DBI action of D0-branes, describing a bound state of a spherical D2-brane with N D0-branes. The quadratic action for small fluctuations is shown to be identical to that obtained from the dual abelian D2-brane DBI action, using the non-commutative geometry of the fuzzy two-sphere. For some of the fields, the linearized equations take the form of solvable Lamé equations. We define a large-N DBI-scaling limit, with vanishing string coupling and string length, and where the gauge theory coupling remains finite. In this limit, the non-linearities of the DBI action survive in both the classical and the quantum context, while massive open string modes and closed strings decouple. We describe a critical radius where strong gauge coupling effects become important. The size of the bound quantum ground state of multiple D0-branes makes an intriguing appearance as the radius of the fuzzy sphere, where the maximal angular momentum quanta become strongly coupled.

Keywords: M(atrix) Theories, Non-Commutative Geometry, D-branes.

Contents

1. Introduction

We consider a time-dependent spherical $D2$-brane system with homogeneous magnetic flux. This is described by a fuzzy sphere solution to the non-abelian action of N D0-branes or equivalently by an abelian $D2$-brane action. The classical solutions have been studied in the context of Matrix Theory and non-abelian DBI in [1 − 6]. Related systems involve $D1 \perp D3$ brane intersections [7 − 9]. Equivalence at the level of classical solutions exists in a large class of examples [5, 10, 11] including higher dimensional fuzzy spheres [12 − 16]. It is natural to explore whether the equivalence at the level of classical solutions extends to an equivalence at the level of quadratic fluctuations.

In this paper we study the fluctuations of the time-dependent $D0$-$D2$ brane system. In section 2, we consider the action for fluctuations using the $D2$-brane action. We find that the result is neatly expressed in terms of the open string variables of [17]. The quadratic action is a (U(1)) Yang-Mills theory with a time-dependent coupling, effective metric and a Θ-parameter. The radial scalar couples to the Yang-Mills gauge field. We analyze the wave equation for the scalar fluctuations and identify a critical radius of the fuzzy sphere where strong coupling effects set in. This radius is different for different values of the angular momentum of the excitations. The fluctuation equation for scalars transverse to the $\mathbb{R}^3$ containing the embedded sphere turns out to belong to the class of solvable Lamé equations. It is very interesting that such an integrable structure appears in a non-supersymmetric context.

In section 3, we obtain the quadratic action for fluctuations on the sphere from the non-abelian symmetrised trace action [18] of N $D0$-branes. We find precise agreement with the action obtained from the $D2$-side. The fact that the commutators $[\Phi_i, \Phi_j]$ contain terms which scale differently with N means that we need to keep $1/N$ terms from commutators of fields. The noncommutative geometry of the fuzzy sphere [19, 20] is reviewed and applied to this derivation. We observe that the mass term for the radial scalar we obtain can also be calculated from the reduced action for the radial variable. This simple calculation is extended to higher dimensional fuzzy spheres and shows similar qualitative features.

In section 4, we describe a DBI scaling limit, where $N \to \infty$, $g_s \to 0$ and $\ell_s \to 0$ keeping fixed the quantities $L = \ell_s \sqrt{\pi N}$, $\tilde{g}_s = g_s \sqrt{N}$ along with specified radius variables and gauge coupling constants. In this limit, the non-linearities of the gauge coupling, which have a square root structure coming from the DBI action, survive. We discuss the physical meaning of this scaling and its connection with the DKPS limit [21], which is important in the BFSS Matrix Model proposal for M-theory [22].

In section 5, we conclude with a discussion of some of the issues and avenues related to the fluctuation analysis of the collapsing $D0$-$D2$ system.

2. Yang-Mills type action for fluctuations

When the spherical membrane is sufficiently large, we may use the Dirac-Born-Infeld (DBI) action to obtain a small fluctuations action about the time dependent solution of [3]. The DBI action is given by

$$-\frac{1}{4\pi^2 g_s \ell_s^3} \int dt\, d\theta\, d\phi \sqrt{-\det(h_{\mu\nu} + \lambda F_{\mu\nu})}\,, \tag{2.1}$$

where $\lambda = 2\pi\ell_s^2$; $h_{\mu\nu}$ is the induced metric on the brane and $F_{\mu\nu}$ describes the gauge field strength on the membrane. The gauge field configuration on the brane consists of a uniform background magnetic field, $B_{\theta\phi} = N \sin\theta/2$, and the fluctuations $f_{\mu\nu}$: $F_{\mu\nu} = (B + f)_{\mu\nu}$. The background magnetic field results from the original N $D0$-branes, which dissolve into uniform magnetic flux inside the $D2$-brane.

To quadratic order in the fluctuations, the action will involve a Maxwell field coupled together with a radial scalar field controlling the size and shape of the membrane. The parameters of this theory will be time-dependent because we are expanding about a time dependent solution to the equations of motion. For the radial field we write $\tilde{R} = R + \lambda(1 - \dot{R}^2)^{1/2}\chi(t, \theta, \phi)$, where R satisfies the classical equations of motion and χ describes the fluctuations. The normalization is chosen for later convenience. We also take into consideration scalar fluctuations in the directions transverse to the $\mathbb{R}^3$ containing the embedded S^2 of the brane worldvolume, described by six scalar fields $\lambda\ \xi_m(t, \theta, \phi)$.

Using the equations of motion, we have that the background field R satisfies the following conservation law equation [3]

$$1-\dot{R}^2=\frac{R^4+N^2\lambda^2/4}{R_0^4+N^2\lambda^2/4}=\frac{R^4+L^4}{R_0^4+L^4}\,. \tag{2.2}$$

We have introduced the physical length L defined by

$$L^2=\frac{N\lambda}{2}\,, \tag{2.3}$$

which simplifies formulas and plays an important role in the scaling discussion of section 4. Here R_0 can be thought of as the initial radius of the brane at which the collapsing rate $\dot{R}$ is zero. The solution $R(t)$ to (2.2) decreases from R_0 to zero, goes negative and then oscillates back to its initial value. It was argued, using the $D0$-brane picture [5], that the physical radius R_{phys} should be interpreted as the modulus of R. Hence this is a periodic collapsing/expanding membrane, which reaches zero size and expands again. The finite time of collapse is given by

$$\bar{t}=c\frac{\sqrt{R_0^4+L^4}}{R_0}\,, \tag{2.4}$$

where the numerical constant c is given by $K(1/\sqrt{2})/\sqrt{2}$, with K a complete elliptic integral.

To leading (zero) order in the fluctuations, the induced metric $h_{\mu\nu}$ on the brane is given by

$$ds^2=-(1-\dot{R}^2)dt^2+R^2(d\theta^2+\sin^2\theta d\phi^2). \tag{2.5}$$

From the form of the induced metric we see that the proper time T measured by a clock co-moving with the brane is related to the closed string frame time t by a varying boost factor

$$dt=\frac{dT}{\sqrt{1-\dot{R}^2}}. \tag{2.6}$$

So an observer co-moving with the collapsing brane concludes that the collapse is actually occurring faster. In terms of proper time, the metric takes the form of a closed three-dimensional Robertson-Walker cosmology

$$ds^2=-dT^2+R^2(T)(d\theta^2+\sin^2\theta d\phi^2) \tag{2.7}$$

with scale factor R. The analogue of the Friedman equation is the conservation law (2.2).

Expanding the DBI action to quadratic order in the fluctuations we obtain the following:

$$S_2=-\int dt\,d\theta\,d\phi\,\frac{\sqrt{-G}}{2g_{YM}^2}\left[\frac{1}{2}G^{\mu\alpha}G^{\nu\beta}f_{\mu\nu}f_{\alpha\beta}+G^{\mu\nu}\partial_\mu\chi\partial_\nu\chi+m^2\chi^2+G^{\mu\nu}\partial_\mu\xi_m\partial_\nu\xi_m\right]. \tag{2.8}$$

The effective metric $G_{\mu\nu}$ seen by the fluctuations is given by

$$ds^2_{\text{open}}=-(1-\dot{R}^2)dt^2+\frac{R^4+L^4}{R^2}d\Omega^2\,. \tag{2.9}$$

As we will see it is precisely the open string metric defined by [17] in the presence of background B-fields. The coupling constant is given by

$$g_{YM}^2 = \frac{g_s}{\ell_s}\frac{\sqrt{R^4+L^4}}{R^2}, \tag{2.10}$$

and the mass of the scalar field is given by

$$m^2 = \frac{6R^2}{(1-\dot{R}^2)\left(R^4+L^4\right)^2}\left(L^4-R^4\right) \ . \tag{2.11}$$

As expected, linear terms in the fluctuations add to total derivatives once we use the equations of motion for the scale factor R.

The set-up here differs from the original set-up of Seiberg/Witten [17] in that we have a non-constant B-field, $B_{\theta\phi} = N\sin\theta/2$. However the basic observation that in the presence of a background magnetic field, the open strings on the brane see a different metric $G_{\mu\nu}$ from the closed string frame metric[1] $h_{\mu\nu}$

$$\begin{aligned} h_{00} &= -(1-\dot{R}^2) \\ h_{\theta\theta} &= R^2 \\ h_{\phi\phi} &= R^2\sin^2\theta \end{aligned} \tag{2.12}$$

continues to be true. The metric $G_{\mu\nu}$ is indeed related to $h_{\mu\nu}$ by

$$G_{00} = h_{00}, \ \ G_{ab} = h_{ab} - \lambda^2(Bh^{-1}B)_{ab} \tag{2.13}$$

or

$$G_{\mu\nu} = h_{\mu\nu} - \lambda^2(Bh^{-1}B)_{\mu\nu}. \tag{2.14}$$

The open string metric (2.9) is qualitatively different from the closed string metric. Despite the fact that the original induced metric $h_{\mu\nu}$ becomes singular when the brane collapses to zero size, the open string metric $G_{\mu\nu}$ is never singular. To see this, let us compute the area of the spherical brane in the open string frame. This is given by

$$A = 4\pi\left(R^2 + \frac{L^4}{R^2}\right). \tag{2.15}$$

As R varies, this function has a minimum at $R = L$, at which $A_{\rm min} = 4\pi N\lambda$ and the density of $D0$-branes is precisely at its maximum $1/4\pi\lambda$; that is, of order one in string units. Effectively, the open strings cannot resolve the constituent $D0$-branes at distance scales shorter than the string length.

The coupling constant can be expressed as $g_{YM}^2 = G_s\ell_s^{-1}$, where

$$G_s = g_s\left(\frac{\det G_{\mu\nu}}{\det(h_{\mu\nu}+\lambda B_{\mu\nu})}\right)^{1/2} = g_s\frac{\sqrt{R^4+L^4}}{R^2}. \tag{2.16}$$

[1]More precisely, the metric $h_{\mu\nu}$ is induced on the brane due to its embedding and motion in the background flat closed string geometry. Distances on the brane defined by using $h_{\mu\nu}$ are also those measured by closed string probes. Thus we shall call $h_{\mu\nu}$ the 'closed string metric'.

So as R decreases, the open strings on the brane eventually become strongly coupled.

There is also a time-dependent vacuum energy density

$$S_0 = -\frac{1}{4\pi^2 g_s \ell_s^3} \int dt\, d\theta\, d\phi \sqrt{-G} \frac{R^2}{\sqrt{R^4 + L^4}}. \tag{2.17}$$

This vacuum energy density can be interpreted as the effective tension of the brane in the open string frame. In terms of the $D2$-brane tension $T_0 = 1/4\pi^2 \ell_s^3$, this is given by

$$T_{\text{eff}} = T_0 \frac{R^2}{\sqrt{R^4 + L^4}}. \tag{2.18}$$

We see that the brane becomes effectively tensionless as $R \to 0$. This is another indication that the theory eventually becomes strongly coupled. The mass of the scalar field χ is a measure of the supersymmetry breaking scale of the theory. Supersymmetry is broken because the brane is compact: the mass tends to zero as $R \to \infty$.

There is a linear term

$$S_1 = \frac{1}{2\lambda^2} \int dt\, d\theta\, d\phi \frac{\sqrt{-G}}{g_{YM}^2} \Theta^{ab} f_{ab}, \tag{2.19}$$

which is a total derivative, and can be dropped if we restrict to gauge fields of trivial first Chern class. It is noteworthy that the open string Θ parameter, given by the standard formulas in terms of closed string frame parameters [17], is precisely what appears here,

$$\Theta^{ab} = \lambda \left(\frac{1}{h + \lambda B} \right)_A^{ab}. \tag{2.20}$$

In terms of R this is given by

$$\Theta^{\theta\phi} = -\frac{2}{N} \frac{L^4}{(R^4 + L^4) \sin\theta}. \tag{2.21}$$

The interpretation of Θ as a non-commutativity parameter will be made more clear in section 4. Notice that this attains its maximum value as $R \to 0$, at which point $\Theta \sim 2/N \sin\theta$ being equal to the inverse background magnetic field.

In addition, there is a non-zero mixing term between the field strength $f_{\mu\nu}$ and the scalar field χ to quadratic order in the fluctuations. This is given by

$$\begin{aligned} S_{\text{int}} &= -\int dt\, d\theta\, d\phi \frac{\sqrt{-G}}{\lambda g_{YM}^2} \frac{2R^3}{\sqrt{1 - \dot{R}^2} \left(\frac{\lambda^2 N^2}{4} + R^4 \right)} \chi \Theta^{ab} f_{ab} \\ &= -\int dt\, d\theta\, d\phi \frac{\sqrt{-G}}{L^2 g_{YM}^2} \frac{R^3}{\sqrt{1 - \dot{R}^2}\,(L^4 + R^4)} \chi (N\Theta^{ab}) f_{ab}. \end{aligned} \tag{2.22}$$

The second line makes it clear that this term is of order one if we consider the physical scaling limit[2] $\ell_s \to 0$, $N \to \infty$, $g_s \to 0$ while keeping R and L fixed. Therefore, it

[2] More on that in section 4.

is comparable to the other terms appearing in the fluctuation analysis. In performing various integrations by parts we have made extensive use of the fact that the combination $(\sqrt{-G}\Theta^{\theta\phi})/g_{YM}^2$ is given by

$$\frac{\sqrt{-G}}{g_{YM}^2}\Theta^{\theta\phi} = -\frac{\lambda^2 N\ell_s}{2g_s}\sqrt{\frac{1-\dot{R}^2}{R^4+\frac{\lambda^2N^2}{4}}} = -\frac{\lambda^2 N\ell_s}{2g_s}\frac{1}{\sqrt{R_0^4+L^4}}\,, \tag{2.23}$$

which is time-independent.

Thus, in the open string frame, the effective metric and non-commutativity parameter are well behaved all the way through the evolution of the brane. The coupling constant diverges as $R \to 0$. From the point of view of open string matter probes on the brane, the sphere contracts to a finite size and then expands again as can be seen from eq. (2.15). But the expansion results eventually in a strongly coupled phase.

The 'open string' parameters $G_{\mu\nu}$, G_s and Θ appearing in the above action are the ones which more naturally would appear in the description of the brane degrees of freedom in terms of non-commutative field variables. We shall show in the next sections how such a description is realized if we replace the smooth membrane configuration (and the uniform background magnetic field) with a system of N $D0$-branes, and re-derive the effective action for the fluctuations from the non-abelian DBI action of the $D0$-brane system in the large-N limit. In the $D0$-brane description the non-commutative variables are $N \times N$ matrices; alternatively, the non-commutative variables can be expressed in terms of functions on a fuzzy sphere whose coordinates are non-commutative [20].

One may turn off the scalar fluctuations χ and consider only fluctuations of the gauge field on spherical branes. In this set up one has a continuum fluid description of the $D0$-branes on the collapsing brane. Indeed the gauge invariant field strength $F_{\mu\nu}$ describes the density and currents of the particles.[3] This continuum description eventually breaks down for two reasons: Firstly the non-commutativity parameter increases, indicating that the fuzziness in area spreads over larger distances. Secondly the gauge field fluctuations become strongly coupled.

2.1 Strong coupling radius

Let us now determine the size of the brane at which the strong coupling phenomenon appears. First notice that the coupling constant g_{YM}^2 is dimensionful, with units of energy. Thus the dimensionless effective coupling constant is given by $g_{YM}^2/E_{\text{proper}}$, where E_{proper} is a typical proper energy scale of the fluctuating modes. The dependence of the effective coupling constant on the energy reminds us that in $2+1$ dimensions the Yang Mills theory is weakly coupled in the ultraviolet and strongly coupled in the infrared. Because of the spherical symmetry of the background solution, angular momentum is conserved including interactions. Thus as the brane collapses, we may determine the relevant proper energy scale in terms of the angular momentum quantum numbers characterizing the fluctuating modes.

[3] Such fluid descriptions are given in the brane constructions of [25, 26].

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To this end, let us examine the massless wave equation, as it arises for example for the transverse scalar fluctuations

$$\partial_\mu \left(\frac{\sqrt{-G}}{g_{YM}^2} G^{\mu\nu} \partial_\nu \xi \right) = 0. \tag{2.24}$$

In terms of angular momentum quantum numbers, this becomes

$$\frac{1}{(1-\dot{R}^2)} \partial_t^2 \tilde{\xi} + \frac{R^2 l(l+1)}{(R^4+L^4)} \tilde{\xi} = 0 \,, \tag{2.25}$$

where we have set $\xi = \tilde{\xi}(t) Y_{lm}$ with Y_{lm} being the appropriate spherical harmonic.

The proper energy is given approximately by

$$E_{\text{proper}} \sim \frac{R\sqrt{l(l+1)}}{\sqrt{(R^4+L^4)}}. \tag{2.26}$$

As the brane collapses the wavelength of massless modes is actually red-shifted! This is essentially because of the form of the effective open string metric.

Now we let the brane collapse to a size $R \ll N^{1/2}\ell_s$. At smaller values of the radius the effective coupling constant becomes

$$g_{\text{eff}}^2 \sim \frac{g_s N^2 \ell_s^3}{R^3 \sqrt{l(l+1)}}. \tag{2.27}$$

Clearly this becomes of order one when R approaches the strong coupling radius R_s

$$R_s = g_s^{1/3} \ell_s \left(\frac{N^2}{\sqrt{l(l+1)}} \right)^{1/3} = L \left(\frac{g_s \sqrt{N}}{\sqrt{l(l+1)}} \right)^{1/3} \,. \tag{2.28}$$

Notice the appearance of $\ell_{11} = g_s^{1/3}\ell_s$, the characteristic scale of Matrix Theory. For l close to the cutoff N, $R_s \sim N^{1/3}\ell_{11}$, which is the estimated size of the quantum ground state of N $D0$-branes [27, 28]. In general R_s involves an effective N given by $N_{\text{eff}} \sim N^2/\sqrt{l(l+1)}$. We shall discuss these special values of the radius in more detail when we describe the membrane after taking various interesting limits for the parameters appearing in (2.28).

The coupling constant of the theory (2.8) is time dependent. We can instead choose to work with a fixed coupling constant absorbing the time-dependence solely in the effective metric if we perform a suitable conformal transformation. By defining $\tilde{G}_{\mu\nu} = \Lambda G_{\mu\nu}$, the gauge field kinetic term gets multiplied by a factor of $\Lambda^{1/2}$. Then we can re-define the coupling constant: $\tilde{g}_{YM}^2 = g_{YM}^2/\sqrt{\Lambda}$. The conformal transformation requires also suitable re-scalings of the fields χ and ξ_m as well as appropriate redefinitions of the various dimensionful parameters of the theory such as m^2 and the non-commutativity parameter Θ^{ab}.

Choosing $\Lambda = (L^4+R^4)/R^4$, the transformed coupling becomes

$$\tilde{g}_{YM}^2 = g_s \ell_s^{-1}, \tag{2.29}$$

and so it is time independent. The open string metric in this frame is still non singular. However, the relevant dimensionless coupling is still the effective coupling $g^2_{\rm eff}$, eq. (2.27), which for small radii remains large. The effect of the conformal transformation gets rid of the time dependence in the coupling constant but also red-shifts $E_{\rm proper}$ by a factor of $\Lambda^{-1/2}$. Therefore, we cannot escape the strong coupling regime in this fashion.

2.2 Overall transverse fluctuations and exactly solvable Schrödinger equation

Another interesting feature of (2.25) is that it is an integrable problem. Using $(1-\dot{R}^2) = (R^4+L^4)/(R_0^4+L^4)$ the wave equation becomes

$$\partial_t^2 \tilde{\xi} + l(l+1)\frac{R^2}{R_0^4+L^4}\tilde{\xi} = 0 \,. \tag{2.30}$$

Substituting the solution for the scale factor R, which is known in terms of the Jacobi elliptic function as $R = R_0\, Cn\left(\frac{t\sqrt{2}R_0}{\sqrt{R_0^4+L^4}}, \frac{1}{\sqrt{2}}\right)$, we have

$$\partial_t^2 \tilde{\xi} + l(l+1)\frac{R_0^2}{R_0^4+L^4} Cn^2\left(\frac{t\sqrt{2}R_0}{\sqrt{R_0^4+L^4}}, \frac{1}{\sqrt{2}}\right)\tilde{\xi} = 0 \,. \tag{2.31}$$

In [5] the solution to the classical problem is related to an underlying elliptic curve. For this specific case we can explicitly express the Jacobi-Cn function in terms of Weierstrass-$\wp$ functions of the underlying curve.[4] The following relation is true for this case

$$Cn^2\left(\sqrt{2}u, \frac{1}{\sqrt{2}}\right) = \frac{\wp(u;4,0)-1}{\wp(u;4,0)+1} \,. \tag{2.32}$$

For these specific functions the following identity also holds

$$\wp(u+\omega_3;4,0) = -\frac{\wp(u;4,0)-1}{\wp(u;4,0)+1} \,, \tag{2.33}$$

where ω_3 is the purely imaginary half period of the relevant elliptic curve in its Weierstrass form, and is given by

$$\omega_3 = i\int_0^1 \frac{ds}{\sqrt{4s(1-s^2)}} \,. \tag{2.34}$$

After a re-scaling of time $t = u\sqrt{L^4+R_0^4}/R_0$ we end up with

$$\partial_u^2 \tilde{\xi} + l(l+1) Cn^2\left(u\sqrt{2}, \frac{1}{\sqrt{2}}\right)\tilde{\xi} = \partial_u^2\tilde{\xi} - l(l+1)\wp(u+\omega_3;4,0)\tilde{\xi} = 0 \,. \tag{2.35}$$

This is exactly the g-gap Lamé equation for the ground state of the corresponding one-dimensional quantum mechanical problem, which has solutions in terms of ratios of Weierstrass σ-functions (for an application in supersymmetric gauge theories see for example [30]).

[4]The interested reader can find a discussion of the Jacobi Inversion problem and its relevance to membrane collapse in [5] and references therein. The formulas that we present here can be checked by consulting appendix C of that paper. A complete mathematical review of the Theory of abelian functions can be found in [29].

A related solvable Schrödinger problem arises in the one-loop computation of the euclidean path integral. This requires the computation of the determinant of the operator

$$-\partial_\tau^2 + \frac{R(i\tau)^2}{\sqrt{R_0^4+L^4}} l(l+1), \tag{2.36}$$

where we have performed an analytic continuation $t \to i\tau$. The eigenvalues of the operator are determined by

$$-\partial_\tau^2\tilde{\xi} + \frac{R(i\tau)^2}{\sqrt{R_0^4+L^4}} l(l+1)\tilde{\xi} = \lambda\,\tilde{\xi}\;. \tag{2.37}$$

In [5] it is shown that $R(i\tau) = 1/R(\tau)$ and that $R^2(i\tau) = \wp(\tau-\Omega;4,0)$ where $\Omega = \int_0^1 \frac{ds}{\sqrt{4s(1-s^2)}}$. Hence the eigenvalue equation becomes

$$-\partial_\tau^2\tilde{\xi} + l(l+1)\wp(\tau-\Omega;4,0)\tilde{\xi} = \lambda\,\tilde{\xi}\;, \tag{2.38}$$

where the eigenstates are also obtained in terms of σ-functions.

We postpone a detailed description and physical interpretation of the solutions of (2.32) and (2.38) for future work. It is intriguing that equation (2.32) has appeared in the literature on reheating at the end of inflation [31]. The physical meaning of this similarity, between fluctuation equations for collapsing D0-D2 systems and those of reheating, remains to be found.

3. Action for fluctuations from the zero-brane non-abelian DBI

The non-abelian DBI action for zero branes [18, 32] is given by

$$S = -\frac{1}{g_s\ell_s}\int dt\; STr\sqrt{-\det(M)}\;, \tag{3.1}$$

where

$$M = \begin{pmatrix} -1 & \lambda\partial_t\Phi_j \\ -\lambda\partial_t\Phi_i & Q_{ij} \end{pmatrix} \tag{3.2}$$

and

$$\begin{aligned} Q_{ij} &= \delta_{ij} + i\lambda\Phi_{ij} \\ \lambda &= 2\pi\ell_s^2\;, \end{aligned} \tag{3.3}$$

with the abbreviation

$$\Phi_{ij} = [\Phi_i, \Phi_j]\;. \tag{3.4}$$

The determinant of M, when the only non-zero scalars lie in the $i,j,k \in \{1,2,3\}$ directions, is given by

$$\begin{aligned} -\det M &= 1 + \frac{\lambda^2}{2}\Phi_{ij}\Phi_{ji} - \lambda^2(\partial_t\Phi_i)(\partial_t\Phi_i) \\ &\quad - \frac{\lambda^4}{2}(\partial_t\Phi_k)(\partial_t\Phi_k)\Phi_{ij}\Phi_{ji} + \lambda^4(\partial_t\Phi_i)\Phi_{ij}\Phi_{jk}(\partial_t\Phi_k)\;. \end{aligned} \tag{3.5}$$

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These terms suffice for the calculation of the quadratic action for the fluctuations involving the gauge field and the radial scalar. However, when we include fluctuations for the scalars Φ_m for $m = 4 \dots 9$ we need the full 10×10 determinant. Fortunately, since we will only be interested in contributions up to quadratic order, the relevant terms will only be those of order up to λ^4

$$\frac{\lambda^2}{2}\Phi_{im}\Phi_{mi}\left(1+\frac{\lambda^2}{4}\Phi_{jk}\Phi_{kj}-\partial_t(\Phi_i)\partial_t(\Phi_i)\right)-\lambda^2\partial_t(\Phi_m)\partial_t(\Phi_m)\left(1+\frac{\lambda^2}{2}\Phi_{ij}\Phi_{ji}\right)$$
$$-\frac{\lambda^4}{4}\Phi_{mi}\Phi_{ij}\Phi_{jk}\Phi_{km}+\lambda^4\partial_t(\Phi_i)\Phi_{im}\Phi_{mj}\partial_t(\Phi_j)-\lambda^4\partial^t(\Phi_m)\Phi_{mi}\Phi_{ij}\partial_t(\Phi_j)\ . \tag{3.6}$$

The expansion with terms of order up to λ^8 is given in [10].

The $D2$-brane solution is described by setting $\Phi_i = \hat{R}(t)X_i$, where the matrices X_i generate the N-dimensional irreducible representation of SU(2). By substituting this ansätz into the $D0$-action, we can derive equations of motion which coincide with those derived from the $D2$ DBI-action [3]. In the correspondence we use

$$R^2 = \lambda^2 C(\hat{R})^2\ , \tag{3.7}$$

where C is the Casimir of the representation, $C = N^2$ in the large-N limit. Note that the square root form in the $D0$-action is necessary to recover the correct time of collapse. If we use the $D0$-brane Yang-Mills limit, we get the same functional form of the solution in terms of Jacobi-Cn functions, but the time of collapse for initial conditions where R_0 is large is incorrect. The correct time of collapse increases as R_0 increases toward infinity, whereas the Yang-Mills limit gives a time which decreases in this limit. The need for the square root was realized in the context of spatial solutions $\hat{R}(\sigma)$ which describe $D1 \perp D3$ funnels [9]. We expand around the solution as follows

$$\begin{aligned}\Phi_i &= \hat{R}X_i + A_i\\ A_i &= 2\hat{R}K_i^a A_a + x_i\phi\\ \Phi_m &= \xi_m\ .\end{aligned} \tag{3.8}$$

The decomposition in the second line above will be explained shortly. Throughout this section, we will be working in the $A_0 = 0$ gauge.

3.1 Geometry of fuzzy two-sphere: brief review

We review some facts about the fuzzy sphere and its application in Matrix theories.[5] As before, the X_i's are generators of the SU(2) algebra satisfying

$$[X_i, X_j] = 2i\epsilon_{ijk}X_k\ . \tag{3.9}$$

With this normalization of the generators, the Casimir in the N dimensional irreducible representation is given by $X_iX_i = (N^2-1)$. If we define $x_i = X_i/N$, we see that

$$\begin{aligned}x_ix_i &= 1\\ [x_i, x_j] &= 0\end{aligned} \tag{3.10}$$

[5] See for example [19, 20].

in the large-N limit. Hence, in the large-N limit the x_i's reduce to cartesian coordinates describing the embedding of a unit 2-sphere in $\mathbb{R}^3$. For traceless symmetric tensors $a_{j_1\ldots j_l}$ the functions $a_{j_1\ldots j_l}x_{j_1}\ldots x_{j_l}$ describe spherical harmonics in cartesian coordinates. Since general (traceless) hermitian matrices can be expanded in terms of (traceless) symmetric polynomials of the X_i's, hence in terms of the x_i's,[6] all our fluctuations such as A_i or transverse scalars such as ξ_m become fields on the sphere in the large-N limit. The expansion of A_i is given by

$$A_i = a_i + a_{i;j}x_j + a_{i;j_1j_2}x_{j_1}x_{j_2} + \ldots\ . \tag{3.11}$$

We can write this as $A_i(t,\theta,\phi)$, with the time dependence appearing in the coefficients a_i, $a_{i;j_1\ldots j_l}$ and the dependence on the angles arising from the polynomial of the x_i's. At finite-N, two important things happen: The x_i's become non-commutative and the spectrum of spherical harmonics is truncated at $N-1$. We will be concerned, in the first instance, with the large-N limit.

The action of X_i on the unit normalized coordinates follows from the algebra (3.9)

$$[X_i, x_j] = 2i\epsilon_{ijk}x_k \tag{3.12}$$

and can be rewritten

$$-2i\epsilon_{ipq}x_p\partial_q(x_j)\ . \tag{3.13}$$

So the adjoint action of X_i can be written as

$$\begin{aligned}[X_i,\ \ ] &= -2iK_i = -2i\epsilon_{ipq}x_p\partial_q \\ &= -2iK_i^a\partial_a\ . \end{aligned} \tag{3.14}$$

We have used Killing vectors K_i defined by $K_i = \epsilon_{ipq}x_p\partial_q$, which obey $x_iK_i = 0$. They are tangential to the sphere and can be expanded as $K_i^a\partial_a$, where a runs over θ,ϕ. The components K_i^a have been used in (3.8) to pick out the tangential gauge field components, and the radial component ϕ defined in (3.8) obeys $\phi = x_iA_i$. It is useful to write down the explicit components of K_i. The Killing vectors K_i^a are given by

$$\begin{array}{ll} K_1^\theta = -\sin\phi & K_1^\phi = -\cot\theta\cos\phi \\ K_2^\theta = \cos\phi & K_2^\phi = -\cot\theta\sin\phi \\ K_3^\theta = 0 & K_3^\phi = 1\ . \end{array}$$

Some useful formulas are the following

$$\begin{aligned} K_i^aK_i^b &= \hat{h}^{ab} \\ x_iK_i^a &= 0 \\ K_i^aK_i^b\partial_ax_j\partial_bx_j &= 2 \\ \epsilon_{ijk}x_iK_j^aK_k^b &= \frac{\epsilon^{ab}}{\sin\theta} = \omega^{ab} \end{aligned} \tag{3.15}$$

[6]The latter give the correctly normalized spherical harmonics as we will explain later.

where $\epsilon^{\theta\phi} = 1$. Here $\hat{h}_{ab}$ is the round metric on the unit sphere and ω^{ab} is the inverse of the symplectic form. As a related remark, note that

$$\Theta^{ab} = -\frac{\lambda^2 N}{2(R^4 + \frac{\lambda^2 N^2}{4})}\epsilon_{ijk}x_i K_j^a K_k^b = -\frac{2}{N}\frac{L^4}{R^4 + L^4}\epsilon_{ijk}x_i K_j^a K_k^b \,. \tag{3.16}$$

We will use these formulas to derive the action for the fluctuations A_a, ϕ geometrically as a field theory on the sphere in the large-N limit. We need one more ingredient. The $D0$-brane action is expressed in terms of traces, which obey the SU(2) invariance condition $Tr(\Phi) = Tr[X_i, \Phi]$. This can be used to show that if Φ is expressed as $\Phi = a + a_j x_j + a_{j_1 j_2} x_{j_1} x_{j_2} + \cdots$, then the trace is just Na, i.e. it picks out the coefficient of the trivial SU(2) representation. By using the similar SU(2) invariance property of the standard sphere integral we have

$$\frac{Tr}{N} \quad \rightarrow \quad \frac{1}{4\pi}\int d\theta d\phi \sin\theta \,. \tag{3.17}$$

This relation between traces and integrals makes it clear why we have chosen the cartesian spherical harmonics to be symmetric traceless combinations of $x_{i_1} \ldots x_{i_l} = (X_{i_1} \ldots X_{i_l})/N^l$. Such spherical harmonics obey

$$\int d\Omega Y_{lm} Y_{l'm'} = \frac{Tr}{N} Y_{lm} Y_{l'm'} = \delta_{ll'}\delta_{mm'}$$

and are the appropriate functions to appear in (3.11).

We make some further general remarks on the calculation, before stating the result for the action obtained from the $D0$-brane picture. Note that the last term in the expansion of the determinant (3.5) gives zero when we evaluate it on the ansätz $\Phi_i = \hat{R}X_i$ used to obtain the solution, but it becomes non-trivial in calculating the action for the fluctuations $\Phi_i = \hat{R}X_i + A_i$. The zero appears because the symmetrised trace allows us to reshuffle the X_i with the $[X_i, X_j]$ for example. Using this property and the commutation relations gives the desired zero and hence leads to agreement between the effective actions for the radial variable, as derived from the $D2$-brane picture.

3.2 The action for the gauge field and radial scalar

Using the ansätz (3.8) we have

$$[\Phi_i, \Phi_j] = (\hat{R})^2[X_i, X_j] + \hat{R}[X_i, A_j] + \hat{R}[A_i, X_j] + [A_i, A_j] \,. \tag{3.18}$$

The first term scales like N, the second two terms are of order one in the large-N limit, while the last term is of order $1/N$. The last commutator term is sub-leading in $1/N$ since the x_i's appearing in (3.11) commute in the strict large-N limit, as of (3.10). When computing terms such as the potential term $\sim [\Phi_i, \Phi_j]^2$, it is important to note that there are terms of order one coming from squaring $\hat{R}[X_i, A_j] + \hat{R}[A_i, X_j]$ as well as from the cross terms $(\hat{R})^2[X_i, X_j][A_i, A_j]$. For this reason, the underlying non-commutative geometry of the fuzzy 2-sphere is important in deriving even the leading terms in the dynamics of the fluctuations.

The first term in (3.18) is simplified by using the commutation relations to give $2i(\hat{R})^2\epsilon_{ijk}X_k = 2iN(\hat{R})^2\epsilon_{ijk}x_k$. The second term can be written as $-4i(\hat{R})^2K_i^a\partial_a(K_j^bA_b) - 2iK_i^a\partial_a(x_j\phi)$ using (3.14). We can compute the leading $1/N$ correction arising from the commutator $[A_i, A_j]$ as follows. We can think of the unit normalized, non-commuting coordinates x_i as quantum angular momentum variables. Since their commutator is given by $[x_i, x_j] = (2i\epsilon_{ijk}x_k)/N$, the analogue of $\hbar$ is given by $2/N$, which scales like the inverse of the spin of the SU(2) representation. Thus the large-N limit is equivalent to the classical limit in this analogy, and in this case all matrix commutators $[A, B]$ can be approximated with 'classical' Poisson brackets as follows

$$[A, B] \to \frac{2i}{N}\{A, B\}\,, \tag{3.19}$$

where $\{A, B\}$ is the Poisson bracket defined by

$$\{A, B\} = \omega^{ab}\partial_a A\;\partial_b B\;, \tag{3.20}$$

using the inverse-symplectic form appearing in (3.15). As a check note that $\{x_i, x_j\} = \epsilon_{ijk}x_k$. The commutator $[A_i, A_j]$ is then given by

$$[A_i, A_j] = \frac{2i}{N}\{A_i, A_j\} + O(1/N^2) = \frac{2i}{N}\omega^{ab}\partial_a A_i \partial_b A_j + O(1/N^2). \tag{3.21}$$

Substituting in (3.1) and expanding the square root, keeping up to quadratic terms in the field strength components $F_{ab} = \partial_a A_b - \partial_b A_a$ and $F_{0a} = \partial_t A_a$, we obtain

$$-\int dt\, d\theta\, d\phi\, \frac{\sqrt{-G}}{4g_{YM}^2}G^{\mu\alpha}G^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}\;, \tag{3.22}$$

where the effective metric and coupling constant are the ones appearing in section 2. Hence we have recovered from the D0-brane action (3.1) the first term of (2.8) obtained from the small fluctuations expansion of the $D2$-brane DBI action. We remark that in calculating the quadratic term in the spatial components of the field strength, the last term in (3.5) gives zero, but its contribution is important in getting the correct coefficient in front of F_{0a}^2.

There is a term linear in F_{ab} given by

$$S_1 = \frac{1}{2\lambda^2}\int dt\, d\theta\, d\phi\, \frac{\sqrt{-G}}{g_{YM}^2}r^4\Theta^{ab}F_{ab} \tag{3.23}$$

where r is the dimensionless radius variable, $r = R/L$. This differs from the linear term obtained from the $D2$ DBI action by the r^4 factor, but the whole term is a total derivative. As such it vanishes in the sector where the fluctuations do not change the net monopole charge of the background magnetic field. This is a reasonable restriction to put when analyzing small fluctuations around a monopole configuration.

At first sight we could also have A_a^2 contributions, which would amount to a mass for the gauge field. Such terms coming from $\partial_t\Phi_i[\Phi_i, \Phi_j][\Phi_j, \Phi_k]\partial_t\Phi_k$ cancel among themselves. The contributions from the other three terms of (3.5) cancel each other up to total derivatives, upon expanding the square root and also performing partial integrations in both the spatial and time directions. Here, we need to use the equation of motion for the scale factor R or $\hat{R}$. Some useful formulas are given in the appendix A. It is important to note that this mass term only vanishes if we keep the terms $[X_i, X_j][A_i, A_j]$, which are order one terms obtained by multiplying the order N with the order $1/N$ commutators.

Next we turn to fluctuations involving the scalar field ϕ. The spatial part of the relativistic kinetic term is

$$-\frac{\ell_s}{2g_s}\int dt\, d\theta\, d\phi\, \sin\theta \frac{(2\lambda N\hat{R}^2)\hat{h}^{ab}\partial_a\phi\partial_b\phi}{\sqrt{(1-\lambda^2N^2\dot{\hat{R}}^2)(1+4\lambda^2N^2\hat{R}^4)}} . \tag{3.24}$$

This agrees with the $D2$-calculation (2.8) if we make the natural identification $\phi = (1 - \lambda^2N^2\dot{\hat{R}}^2)^{1/2}\chi$. Following this, we can match the quadratic terms in $\partial_t\phi$ and we find again that we get the same answer as from the $D2$-side. The overall kinetic term is given by

$$-\int dt\, d\theta\, d\phi\, \frac{\sqrt{-G}}{2g_{YM}^2}G^{\mu\nu}\partial_\mu\chi\partial_\nu\chi \tag{3.25}$$

as in (2.8).

For the mass term of χ, we get

$$\begin{aligned}
-\frac{N}{4\pi\ell_s g_s}&\int dt\, d\theta\, d\phi\, \sin\theta \frac{12\lambda^2\hat{R}^2(1-4N^2\lambda^2\hat{R}^4)}{(1+4N^2\lambda^2\hat{R}^4)^{3/2}\sqrt{1-N^2\lambda^2\dot{\hat{R}}^2}}\chi^2 \\
&= -\int dt\, d\theta\, d\phi\, \frac{\sqrt{-G}}{2g_{YM}^2}\frac{6R^2(L^4-R^4)}{(L^4+R^4)^2(1-\dot{R}^2)}\chi^2 .
\end{aligned} \tag{3.26}$$

This agrees with the mass for χ in (2.8). Another thing to note here is that the determinant will also give contributions linear in ϕ and $\partial_t\phi$ and also terms quadratic in the scalar fluctuation of the form $\phi\, \partial_t\phi$. However, upon the expansion of the square root to quadratic order the overall linear factor of ϕ cancels. We recall that we are expecting the latter, since ϕ is a fluctuation around a background which solves the equations of motion. Upon conversion to the χ variable the kinetic term for ϕ will also contribute $\chi\, \partial_t\chi$ terms. Then by integrating by parts and dropping the respective total time derivative terms we end up with the appropriate mass for χ given above.

For the mixing terms between F_{ab} and ϕ, collecting all the relevant terms one gets

$$-\int dt\, d\theta\, d\phi\, \frac{\sqrt{-G}}{g_{YM}^2}\frac{8\hat{R}^3N}{(1+4\lambda^2N^2\hat{R}^4)\sqrt{1-\lambda^2N^2\dot{\hat{R}}^2}}\chi\Theta^{ab}F_{ab} . \tag{3.27}$$

Once more, we get exact agreement with the $D2$-calculation (2.22). Finally the quadratic action for the scalars ξ_m obtained by expanding the terms in (3.6) is easily seen to be

$$-\int dt\, d\theta\, d\phi\, \frac{\sqrt{-G}}{g_{YM}^2}G^{ab}\partial_a\xi_m\partial_b\xi_m , \tag{3.28}$$

which agrees with (2.8).

3.3 Scalar fluctuations for the reduced action

We expect to be able to reach the same results for the scalar fluctuations by just considering the large-N reduced action for the background fields as in [3, 5]

$$S_2 = -\frac{2}{g_s \ell_s \lambda} \int dt \sqrt{1-\dot{R}^2} \sqrt{R^4 + \frac{N^2 \lambda^2}{4}}, \tag{3.29}$$

and consider adding fluctuations $R \to R + \lambda\sqrt{1-\dot{R}^2}\,\chi$ as before. One gets

$$\begin{aligned} S_2^{\text{mass}} &= -\frac{2}{g_s \ell_s \lambda} \int dt \frac{\lambda^2 R^2 (3L^4 - 3R^4)}{2(L^4+R^4)^{3/2}\sqrt{1-\dot{R}^2}} \chi^2 \\ &= \frac{-4L\sqrt{\pi}}{\tilde{g}_s} \int dt \frac{R^2 (3L^4 - 3R^4)}{2(L^4+R^4)^{3/2}\sqrt{1-\dot{R}^2}} \chi^2 \end{aligned} \tag{3.30}$$

the same answer for the mass of the scalar fluctuation as by perturbing the full action (3.1), when written in terms of g_{YM}^2 and $\sqrt{-G}$.

We can make use of this result to check the behavior of the scalar mass for higher even spheres. The reduced action for the fuzzy S^4 is [5]

$$S_4 = -\frac{4}{g_s \ell_s \lambda^2 N} \int dt \sqrt{1-\dot{R}^2} \left(R^4 + \frac{\lambda^2 N^2}{4}\right) . \tag{3.31}$$

Perturbing this will result to a mass

$$\begin{aligned} S_4^{\text{mass}} &= -\frac{4}{g_s \ell_s \lambda^2 N} \int dt \frac{2\lambda^2 R^2 (3L^4 - 5R^4)}{(L^4+R^4)\sqrt{1-\dot{R}^2}} \chi^2 \\ &= -\frac{8\sqrt{\pi}}{\tilde{g}_s L} \int dt \frac{R^2 (3L^4 - 5R^4)}{(L^4+R^4)\sqrt{1-\dot{R}^2}} \chi^2, \end{aligned} \tag{3.32}$$

where we have made use of the appropriate equations of motion.

There is a similar behavior for the S^6. The reduced action is [5]

$$S_6 = -\frac{8}{g_s \ell_s \lambda^3 N^2} \int dt \sqrt{1-\dot{R}^2} \left(R^4 + \frac{\lambda^2 N^2}{4}\right)^{3/2} , \tag{3.33}$$

and the result for the mass

$$\begin{aligned} S_6^{\text{mass}} &= -\frac{8}{g_s \ell_s \lambda^3 N^2} \int dt \frac{12\lambda^2 R^2 (3L^4 - 7R^4)}{\sqrt{L^4+R^4}\sqrt{1-\dot{R}^2}} \chi^2 \\ &= -\frac{48\sqrt{\pi}}{\tilde{g}_s L} \int dt \frac{R^2 (3L^4 - 7R^4)}{\sqrt{L^4+R^4}\sqrt{1-\dot{R}^2}} \chi^2. \end{aligned} \tag{3.34}$$

The physical behavior remains the same for any k: for the pure $N=0$ case the scalar mass squared is negative from the beginning of the collapse all the way down to zero. At finite (large) N there is a transition point which depends on the dimensionality k.

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3.4 $1/N$ correction to the action

The derivation of the the action from the $D0$-brane side can easily be extended to include $1/N$ corrections. The net outcome will be a non-commutative gauge theory, where products are replaced by suitable star products. Two important features have to be noted. It is no longer consistent to assume $x_i K_i^a = K_i^a x_i$. This is because

$$\begin{aligned}
[x_i, K_i^\theta] &= -\frac{2i}{N}\cot\theta \\
[x_i, K_i^\phi] &= 0 \ .
\end{aligned} \tag{3.35}$$

We can instead only assume $x_i K_i + K_i x_i = 0$. We also have a first correction to the Leibniz rule for the partial derivatives

$$\partial_a(FG) = (\partial_a F)G + F(\partial_b G) - \frac{i}{N}(\partial_a \omega^{bc})(\partial_b F)(\partial_c G) \ . \tag{3.36}$$

This is consistent with

$$\partial_a[x_i, x_j] = [\partial_a x_i, x_j] + [x_i, \partial_a x_j] - \frac{i}{N}(\partial_a \omega^{bc})[\partial_b x_i, \partial_c x_j] \ . \tag{3.37}$$

4. Scaling limits and quantum observables

Given the action we have derived from the $D0$ and $D2$-sides, there are several limits to consider so as to describe the physics.

4.1 The DBI-scaling

Consider $g_s \to 0$, $\ell_s \to 0$, $N \to \infty$ keeping fixed

$$R, \ L = \ell_s \sqrt{\pi N}, \ g_s \sqrt{N} \equiv \tilde{g}_s \ . \tag{4.1}$$

In this limit the following parameters appearing in the lagrangian are fixed

$$\begin{aligned}
g_{YM}^2 &= \frac{g_s}{\ell_s}\frac{\sqrt{R^4+L^4}}{R^2} = \frac{\sqrt{\pi}\tilde{g}_s}{L}\frac{\sqrt{R^4+L^4}}{R^2} \\
G_{00} &= \sqrt{1-\dot{R}^2} \\
G_{ab} &= \frac{R^4+L^4}{R^2}\hat{h}_{ab} \ .
\end{aligned} \tag{4.2}$$

We also keep fixed in this limit the energies and angular momenta of field quanta in the theory.

With this scaling all the quadratic terms of the field theory action on S^2 derived from the D2-brane side in (2.8), (2.22), and reproduced in section 3 from the D0-branes, remain fixed. Notice that all terms in (3.5) are also of order one and all of them contribute so as to obtain the small fluctuations action and the parameters of the theory given above. In addition, since in this limit $\ell_s \to 0$, massive open string modes on the branes decouple, and we can neglect higher derivative corrections to the DBI action. Further, since $g_s \to 0$, we expect closed string emission to be negligible. This scaling should be compared to scalings

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studied in Matrix Theory in [21, 27, 33−37]. In the region $R \ll L$, we will consider the relation to the Matrix Theory limit below.

There are several interesting features of the limit (4.1). It allows us to neglect the finite size effects of the quantum $D0$-brane bound state. The quantum field theory we have derived by expanding around the classical solution might be expected to be invalid in the regime where the radius of the sphere reaches the size R_q of the quantum ground state of N $D0$-branes. This has been estimated to be [27, 28]

$$\begin{aligned} R_q &= N^{1/3} g_s^{1/3} \ell_s \\ &= \frac{\tilde{g}_s L}{N^{1/3}} \,. \end{aligned} \tag{4.3}$$

Clearly this is zero in the scaling limit, which gives us reason to believe that the DBI action is valid all the way to $R = 0$.

Another issue is gravitational back-reaction. This can be discussed by comparing the radius of the collapsing object to the gravitational radius of a black hole with the same net charge. This type of argument is used for example in [38] for studying collapsing domain walls in four dimensions. We find that in the scaling limit (4.1), gravitational back-reaction is negligible. To see this consider first the excess energy ΔE of the classical configuration above the ground state energy of N $D0$-branes. For extremal black holes the horizon area is zero. For non-extremal ones, it is directly determined by the excess energy [39]

$$R_h^8 = g_s^{\frac{25}{14}} \ell_s^{\frac{121}{14}} \sqrt{N} (\Delta E)^{\frac{9}{14}}. \tag{4.4}$$

Using

$$\begin{aligned} \Delta E &= \frac{N}{g_s \ell_s} \left(\frac{\sqrt{R_0^4 + L^4}}{L^2} - 1 \right) \\ &= \frac{N^2}{\tilde{g}_s L} \left(\frac{\sqrt{R_0^4 + L^4}}{L^2} - 1 \right) , \end{aligned} \tag{4.5}$$

we find for the horizon radius

$$R_h^8 = N^{\frac{-24}{7}} \tilde{g}_s^{\frac{8}{7}} L^8 \left(\frac{\sqrt{R_0^4 + L^4}}{L^2} - 1 \right)^{\frac{9}{14}} , \tag{4.6}$$

which goes to zero in the large-N limit. This shows that gravitational back-reaction resulting in the formation of non-extremal black holes does not constrain the range of validity of the DBI action in our scaling limit.

Another black hole radius we may compare to is the Schwarzschild radius for an object having energy $N\sqrt{R_0^4 + L^4}/(g_s \ell_s L^2)$, as is the case for our membrane configuration. This comparison is more relevant in the limit $R \gg L$ where the $D0$-brane density is small; in other words the charge density of the relevant black hole is small. In this case we expect the discussion of [38] to be most relevant. The Schwarzschild radius is given by

$R_{sch} = (G_s E)^{1/7}$, or more explicitly

$$R_{sch} = N^{\frac{-3}{7}} L \tilde{g}_s^{\frac{1}{7}} \left(\frac{\sqrt{R_0^4 + L^4}}{L^2} \right)^{\frac{1}{7}} \quad . \tag{4.7}$$

This is also zero in the scaling limit (4.1), and hence does not invalidate the DBI action.

Since R is time-dependent, the parameters of the theory are also time-dependent. We may consider correlators of gauge invariant operators

$$\langle \mathcal{O}(t_1, \sigma_1^a) \mathcal{O}(t_2, \sigma_2^a) \ldots \rangle \tag{4.8}$$

where $\mathcal{O}$ can be for example $Tr(F^2)$ or $Tr(\Phi^2)$, which use the field strength or transverse scalars. For times $t_1, t_2, \ldots$ corresponding via the classical solution to R near R_0, the Yang-Mills coupling is small, and the approximation where non-linearities of the DBI have been neglected is a valid one. So we can compute such correlators perturbatively. When R approaches zero, the Yang-Mills coupling diverges, so we need to use the all-orders expansion of the DBI action. We have not computed the fluctuation action to all orders, but it is in principle contained in the full DBI action.

An interesting observable is $\langle 0|\chi|0\rangle$ which gives quantum corrections to the classical path. In time dependent backgrounds, one can typically define distinct early and late times vacua because positive and negative frequency modes at early and late times can be different. If we set up an early times vacuum in the ordinary manner, and write χ as a linear combination of early times creation and annihilation operators, the one point function of χ in the late times vacuum may be non-zero indicating particle production. The non-trivial relation between in and out-vacua is certainly to be expected for all the fields in the theory, since it is a generic feature of quantum fields in a time dependent background [40]. Recent applications in the decay of unstable branes include [41 − 43].

We have argued that radiation into closed string states is negligible because their coupling constant $g_s \to 0$ in the scaling limit (4.1). In the context of open string tachyon condensation, describing brane decay, the zero coupling limit of closed string emission was shown not to approach zero as naively expected because of a divergence coming from a sum over stringy states [42]. Here we may hope to escape this difficulty because $\ell_s \to 0$ means that the infinite series of massive closed string states decouple and the Hagedorn temperature goes to infinity. Of course in the tachyonic context [42], the limit $\ell_s \to 0$ could not be taken since it would force the tachyon to be infinitely massive as well. To prove that there is no closed string production will require computation of the one-loop partition function in the theory expanded around the solution and showing that any non-vanishing imaginary part obtained in the limit (4.1) can be interpreted in terms of the DBI action (3.1). Such computations in a supersymmetric context are familiar in Matrix Theory. Recent work has also explored the non-supersymmetric context [44].

We have argued that open strings on the membrane eventually become strongly coupled when the physical radius is given by eq. (2.28). This special value for the radius remains fixed in our scaling limit: $R_s \sim (\tilde{g}_s)^{1/3} L$. It can be made arbitrarily small if we take $\tilde{g}_s$ sufficiently small. But for any fixed value of this coupling, however small it is, strong coupling quantum effects are eventually needed to understand the subsequent membrane evolution. Quantum processes may cause the original brane with N units of $D0$-brane charge to split into configurations of smaller charge. However such non-perturbative phenomena should be describable within the full non-abelian $D0$-brane action (3.1).

We can also construct multi-membrane configurations. For example, we can construct m coincident spherical membranes if we start with the non-abelian DBI action of mN $D0$-branes and replace the background values of the matrices Φ_i in (3.8) with the following block-diagonal forms [20]

$$\hat{R}X_i \to \hat{R}X_i \otimes \mathbb{1}_m. \tag{4.9}$$

The fluctuation matrices A_i are replaced by

$$A_i \to \sum_1^{m^2} A_i^\alpha \otimes T^\alpha \tag{4.10}$$

where the $m \times m$ matrices T^α are generators of $\mathrm{U}(m)$. Taking the large-N limit, while keeping m fixed, the action for the fluctuations should result in a non-abelian $\mathrm{U}(m)$ gauge theory on a sphere describing a collection of m coincident spherical $D2$-branes. The field strength of the U(1) part of this gauge group attains a background value corresponding to mN units of flux on the sphere. We expect the effective metric and coupling constant of this theory to be given by the same formulas that we have derived before. Separate stacks of $D2$-branes can be constructed by giving an appropriate vacuum expectation value to one of the transverse scalars; that is, by 'Higgsing' the $\mathrm{U}(M)$ gauge group. The net background magnetic flux should now split appropriately among the separate stacks. Within this set-up, one can study non-perturbative instanton processes that result into transferring of $D0$-branes from one membrane stack to another, as in [45]. The effective dimensionless coupling of such processes is given approximately by $g_{YM}^2/\langle\phi\rangle$, where $\langle\phi\rangle$ is the relevant Higgs VEV. When the branes are large, that is $R > L$, this coupling can be kept small if we take $\tilde{g}_s$ small, and such processes are exponentially suppressed. But when the radius becomes small, the theory becomes strongly coupled and such non-perturbative processes become relevant.

4.2 The D0 Yang Mills (Matrix theory) limit

In this limit, we take $R/L = r$ as well as r_0 to be small. We will show how the effective action for the fluctuations in this regime can be derived from the BFSS Matrix Model [22]. Earlier work on this model appears in [23, 24].

The effective parameters of the theory are $G_{\mu\nu}$, G_s and Θ^{ab}. In terms of the dimensionless radius variable r, these are given by

$$\begin{aligned}
G_{00} &= -\sqrt{\frac{r^4+1}{r_0^4+1}}, \quad G_{ab} = \frac{N\lambda}{2}\left(r^2 + \frac{1}{r^2}\right)\hat{h}_{ab} \\
G_s &= g_s \frac{\sqrt{r^4+1}}{r^2} \\
\Theta^{ab} &= -\frac{2\epsilon^{ab}}{N(1+r^4)\sin\theta}\ .
\end{aligned} \tag{4.11}$$

When $r, r_0 \ll 1$, these take the following 'zero slope' form [17, 46]

$$\begin{aligned}
&G_{00} \to \tilde{G}_{00} = 1, \quad G_{ab} \to \tilde{G}_{ab} = -\lambda^2 (Bh^{-1}B)_{ab} = \frac{N\lambda}{2r^2}\hat{h}_{ab} \\
&G_s \to \tilde{G}_s = g_s \det(\lambda B h^{-1})^{\frac{1}{2}} = \frac{g_s}{r^2} \\
&\Theta^{ab} \to \tilde{\Theta}^{ab} = (B^{-1})^{ab} = -\frac{2\epsilon^{ab}}{N\sin\theta} \,.
\end{aligned} \tag{4.12}$$

Notice that the rate of collapse $\dot{R}$ is given by

$$\dot{R}^2 = r_0^4 - r^4 \tag{4.13}$$

in this regime. In particular, this remains small throughout the collapse of the brane.

We can 'derive' these zero slope parameters from the effective action of the constituent D0-branes in the small-r regime. The background fields scale as

$$\begin{aligned}
\Phi_i &= \hat{R}X_i = \left(\frac{r}{2L}\right) X_i \\
\partial_t \Phi_i &= (\partial_t \hat{R}) X_i \sim \left(\frac{\sqrt{r_0^4 - r^4}}{2L^2}\right) X_i.
\end{aligned} \tag{4.14}$$

We assume a similar scaling behavior for the fluctuations $A_i = 2\hat{R}K_i^a A_a + x_i\phi$ and their time derivatives $\partial_t A_i$ in the small-r regime. That is, we take the gauge field A_a to be of order one while the radial fluctuations $x_i\phi$ to be at most of the order r/L in magnitude. Similarly, the velocity fields $\partial_t A_a$ and $x_i \partial_t \phi$ are required to be of order r/L and r^2/L^2 respectively. This is a reasonable requirement for the behavior of the fluctuations so as to keep them smaller or at least comparable to the background values of the fields. Then the full fields Φ_i and their time derivatives are sufficiently small in the small-r regime, and the $D0$-brane effective action (3.1) takes the form of a $0+1$ dimensional Yang-Mills action:

$$S = \frac{(2\pi)^2 \ell_s^3}{g_s} \int dt \left[Tr\left(\frac{1}{2}\partial_t\Phi_i\partial_t\Phi_i + \frac{1}{4}[\Phi_i, \Phi_j]^2\right) - \frac{N}{\lambda^2}\right]. \tag{4.15}$$

The second and third terms in (3.5) scale as r^4 in the limit, while the last two terms as higher powers of r. In the small-r regime, we can neglect the last two terms of (3.5) and expand the square root of the DBI action dropping higher powers of r. We end up with an action that is quadratic in the time derivatives of the fields and quartic in the fields themselves.[7] Roughly speaking, in this regime each $D0$-brane is moving slowly enough so that the non-relativistic, small velocity expansion of the DBI lagrangian can be applied ending up with (4.15). This expansion is valid if we choose the initial radius parameter r_0 to be small enough, or the initial physical radius to satisfy $R_0 \ll L$. Essentially the Yang Mills regime is valid when the effective separation of neighboring D0-branes is smaller than

[7] A similar expansion can be consistently carried out for the fields Φ_m that are transverse to the $\mathbb{R}^3$ where the membrane is embedded.

the string scale throughout the collapse of the brane. Finally, in this regime the equation of motion for the scale factor is given by

$$\begin{aligned} \ddot{\hat{R}} + 8\hat{R}^3 &= 0 \\ \ddot{r} + \frac{2}{L^2} r^3 &= 0. \end{aligned} \tag{4.16}$$

Setting $\Phi_i = \hat{R}X_i + A_i$, we can determine a matrix model for the fluctuating fields A_i. This matrix model is equivalent to a non-commutative U(1) Yang Mills theory on a fuzzy sphere [20]. This correspondence maps hermitian matrices to functions on the sphere, and replaces the matrix product with a suitable non-commutative star product. To see how the non-commutative gauge fields arise, we examine the transformation of the fluctuating matrices A_i under time independent infinitesimal U(N) gauge transformations, which are symmetries of the action (4.15). Under such a gauge transformation, the matrices Φ_i and A_i transform as follows

$$\begin{aligned} \delta_\lambda \Phi_i &= i[\lambda, \Phi_i] \\ \delta_\lambda A_i &= -i\hat{R}[X_i, \lambda] + i[\lambda, A_i], \end{aligned} \tag{4.17}$$

with λ an $N \times N$ hermitian matrix. Using equation (3.14), the corresponding function on the sphere transforms as[8]

$$\delta_\lambda A_i = 2\hat{R}K_i^a \star \partial_a \lambda + i(\lambda \star A_i - A_i \star \lambda), \tag{4.18}$$

where λ is now taken to be a local function on the sphere. Thus we end up with a U(1) non-commutative gauge transformation. The gauge covariant field strength is given by

$$F_{ij} = i\hat{R}[X_i, A_j] - i\hat{R}[X_j, A_i] + i[A_i, A_j] + 2\hat{R}\epsilon_{ijk}A_k = i[\Phi_i, \Phi_j] + 2\hat{R}\epsilon_{ijk}\Phi_k. \tag{4.19}$$

The last equation makes gauge covariance manifest. The field strength F_{ij} is zero when the fluctuations are set to zero, while the commutator $[\Phi_i, \Phi_j]$ attains a background expectation value given by $\hat{R}^2[X_i, X_j]$.

In the commutative limit, the non-commutative gauge transformations (4.18) reduce to ordinary local U(1) gauge transformations. As we already discussed, this is equivalent to a large-N limit. Decomposing $A_i = 2\hat{R}K_i^a A_a + x_i\phi$ we see that in the commutative limit, the tangential fields A_a transform as the components of a gauge field on the sphere, $\delta_\lambda A_a = \partial_a \lambda$, while the transverse field ϕ as a scalar. The full non-commutative gauge transformation (4.18) though mixes ϕ and the vector field A_a [20]; this is another manifestation of the fuzziness of the underlying space.

It is easy to see that in the commutative limit the field strength reduces to

$$F_{ij} \to 4\hat{R}^2 K_i^a K_j^b F_{ab} + 2\hat{R}(x_j K_i^a - x_i K_j^a)\partial_a \phi - 2\hat{R}\epsilon_{ijk}x_k\phi. \tag{4.20}$$

[8] We do not use different notation to distinguish the $N \times N$ hermitian matrices from their corresponding functions on the sphere. We hope the distinction is made clear from the context.

The deformation arising from the underlying non-commutativity comes from the commutator piece $i[A_i, A_j]$ in (4.19). Up to the order of $1/N$, this deformation is given by (3.21), and can be rewritten as

$$i[A_i, A_j] = \tilde{\Theta}^{ab}\partial_a A_i \partial_b A_j + O(\tilde{\Theta}^2). \tag{4.21}$$

We conclude immediately that the underlying non-commutativity parameter is $\tilde{\Theta}$.

We can expand the $D0$-brane Yang Mills action (4.15) to quadratic order in the fluctuations in the large-N limit. Having established the equivalence of the full $D0$-DBI action with the $D2$-brane action to this order in the fluctuations, all we need to do is to replace the effective metric, coupling constant and non-commutativity parameter with their 'zero slope' values (4.12). Of course, one can carry out the expansion directly using the action (4.15) and verify that the parameters of the theory in this regime are indeed given by $\tilde{G}_{\mu\nu}$, $\tilde{g}^2_{YM}$ and $\tilde{\Theta}$. The mass of the scalar field χ defined above eq. (3.25) is given by

$$m^2 = \frac{6r^2}{L^2} \tag{4.22}$$

in this regime and it is positive. Finally, the mixing term becomes

$$-\int dt\, d\theta\, d\phi\, \frac{\sqrt{-\tilde{G}}}{\tilde{g}^2_{YM}} \frac{r^3}{L^3}\chi(N\tilde{\Theta}^{ab})F_{ab}. \tag{4.23}$$

It is important to realize that non-linearities in the equations of motion, arising from interaction terms of higher than quadratic order in the non-relativistic lagrangian (4.15), are all suppressed by factors of $1/N$. From the point of view of the U(1) non-commutative field theory on the fuzzy sphere, all interaction terms arise from the non-commutative deformation of the field strength (4.19) and they end up being proportional to powers of $\tilde{\Theta}$. It is easy then to see that non-linearities become important at angular momenta of order $l \sim N^{1/2}$ where $\tilde{\Theta}^{ab}\partial_a \otimes \partial_b$ is of order one. This fact was also emphasized in the analysis of [47]. From (2.28) then we see that such angular momentum modes become strongly coupled when

$$R \sim \ell_{11} N^{1/2} \tag{4.24}$$

or

$$r \sim {g_s}^{1/3}. \tag{4.25}$$

Roughly, the strong coupling phenomenon occurs when in the closed string frame each $D0$-brane occupies an area of order ℓ^2_{11}, smaller than ℓ^2_s.

In the scaling limit (4.1), the eleven dimensional Planck length tends to zero like $N^{-2/3}$ and the strong coupling radius (4.24) goes to zero. Thus in the limit (4.1) the evolution of such small branes, described by the $D0$-brane Yang Mills action (4.15), can be treated classically throughout the collapse of the brane. We can alternatively take a different scaling limit so as to probe the short eleven dimensional Planck scale, which sets the distance scale at which strong coupling quantum phenomena occur in our system in the non-relativistic regime.

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We can take $g_s \to 0$ keeping R and ℓ_{11} fixed, and also N fixed and large. In this limit $L \to \infty$ like $g_s^{-1/3}$, so that r and also r_0 are small. The physical field variables $\lambda\Phi_i \sim RX_i/N$, and so they remain fixed in this limit. The same is true for their conjugate momenta. At the same time each individual $D0$-brane is getting very heavy since $m_{D0} = 1/g_s\ell_s \sim g_s^{-2/3}/\ell_{11}$. Hence the $D0$-branes are slowly moving in this limit. This limit is the famous DKPS limit [21, 27] in which the short distance scale probed by the $D0$-branes is the eleven dimensional Planck scale. Closed strings decouple from the brane system. The same is true for excited massive open strings on the branes. This is because the energy of the fluctuating massless open string states is much smaller than the mass of excited open string oscillators in the limit [21, 27] and so massive open strings cannot get excited. Finally, in the BFSS limit [22] where the eleven circle radius is decompactified, the membrane we constructed is just a boosted spherical M-theory membrane.

The strong coupling phenomenon above occurs at a physical radius which is bigger than the size of the bound quantum ground state (of the N $D0$-branes) by a factor of $N^{1/6}$. However angular momentum modes of order the cutoff N become strongly coupled when the physical radius R becomes comparable to the size of the ground state $\ell_{11}N^{1/3}$ as can be seen from (2.28). It is interesting that this scale which is expected to emerge from a complicated ground state solution of the $D0$-brane Yang-Mills hamiltonian also appears in the analysis of the linearized fluctuations of fuzzy spheres.

There is yet another simple way to see the ℓ_{11} length scale. It involves the application of the Heisenberg uncertainty principle to the reduced radial dynamics. The momentum conjugate to R coming from the reduced action (3.29) is

$$\Pi_R = \frac{\sqrt{R_0^4 - R^4}}{g_s\ell_s^3\pi} \,. \tag{4.26}$$

With $(\Delta R)\,\Pi_R > \hbar$ and $\hbar \sim 1$, we get

$$(\Delta R) > \frac{g_s\ell_s^3\pi}{\sqrt{R_0^4 - R^4}} \,. \tag{4.27}$$

Evaluating the uncertainty at $R = 0$ and assuming the whole trajectory lies within the quantum regime, i.e. $R_0 \sim \Delta R$, we obtain a critical value for the initial radius $R_0 \sim R_c$ where $R_c \sim g_s^{1/3}\ell_s$, which is the eleven dimensional Planck scale. This simple analysis does not detect the $N^{1/3}$ factor that appears in the more complete analysis above.

The above discussion has focused on the region where R is much smaller than L. The region of $R \gg L$ or equivalently $L = \ell_s\sqrt{\pi N} \to 0$ is also of interest. In the strict $N = 0$ limit we have a $D2$-brane without $D0$-brane charge. The negative sign of the mass of the field χ that appears in (3.30) for $R > L$ also appears in the problem of fluctuations around the pure $D2$-brane solution. This negative sign indicates that the zero mode of the field χ is tachyonic in this regime. When R_0 is larger than L, the tachyonic mass naively causes an exponential growth for the zero mode of the fluctuation χ. At this point, higher order corrections to the action involving the zero mode would become significant. However, the reduced action for the scalar dynamics has no exponentially growing solutions. This means that higher order terms stop this exponential growth. In fact, as R crosses L, the sign of the mass changes and we go into an oscillatory phase. This transition is reminiscent of a similar transition which occurs in the equation for fluctuations in inflationary scenarios, see for example [48]. In the case $R_0 \leq L$ the time evolution of the radial fluctuation does not encounter the tachyonic region.

4.3 Mixing with graviton scattering states

The key observable in the BFSS Matrix theory limit is the scattering matrix of $D0$-brane bound ground states made of $N_1, N_2, \dots N_i$ $D0$-branes, where N_i are all large. Since these interactions are governed by ℓ_{11}, which goes to zero in the scaling limit (4.1), such interactions among such states become irrelevant. However a simple estimate suggests that these states can mix with the fuzzy sphere states. Consider an SU(2) representation of spin j with $N = 2j+1$. Consider also matrices

$$U_\pm = \begin{pmatrix} 0 & 0 \\ 0 & b \pm iv \end{pmatrix} .$$

The diagonal blocks are of size $N_1 \times N_1$ and $(N - N_1) \times (N - N_1)$. There are also the standard $N \times N$ SU(2) matrices J_+, J_-, which act in this representation. In the fuzzy sphere configuration we set $X_\pm \equiv (X_1 \pm iX_2) = J_\pm$ while in the scattering configuration we set $X_\pm = U_\pm$. We calculate $Tr([J_+, U_-][J_-, U_+])$ and find this proportional to $\lambda^2(\hat{R})^2 N(N - N_1)$. If N_1 is a finite fraction of N then this goes like $\lambda^2(\hat{R})^2 N^2 \sim (\hat{R})^2 L^4$ in the large-N limit, which is of the same order as the terms in the quadratic action for the fluctuations we have computed. This indicates that the collapsing membrane can undergo transitions to these scattering states and conversely the scattering states can give rise to membranes.

5. Discussion and outlook

The discussion of the scaling limit in section 4 is very reminiscent of similar scaling limits in the context of BFSS Matrix Theory [22] and the AdS/CFT duality [49]. The difference is that here we are keeping the non-linearities coming from the non-abelian $D0$-brane action (3.1). This action is of course less understood than the $0+1$ SYM of the BFSS Matrix Model or the $3+1$ SYM of the canonical AdS/CFT correspondence. For example a completely satisfactory supersymmetric version has yet to be written down, although some progress on this has been discussed in [18]. However it is significant that our scaling discussion of section 4.1 highlights the fact that the appropriate supersymmetrised non-abelian DBI action should provide a complete quantum mechanical description of the collapsing $D0$-$D2$ system.

This may appear somewhat surprising, but we will argue not unreasonable. When R is close to R_0 we have a Yang-Mills action at weak coupling, and quantum correlation functions can be computed in a weak coupling expansion. In the strict large-N limit, the Yang-Mills theory is commutative, while $1/N$ corrections amount to turning the background sphere into a non-commutative sphere. When the correlation functions are localized in regions where the radius is small, the Yang-Mills coupling is large and non-linearities in the fields become important. There must be a quantum mechanical framework which

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provides the continuation of the correlators to this region. Since ℓ_s has been taken to zero, massive string modes decouple and the only degrees of freedom left to quantize are those that appear in (3.1). String theory loops degenerate to loops of the fields in this action. The conjecture suggested by these arguments is that the fate of the collapsing $D0 - D2$ system in the zero radius region, and in the regime of parameters of section 4, is contained in the quantum version of the supersymmetrised non-abelian DBI of (3.1). We have outlined a framework for calculating quantum corrections to the classical bouncing path, and discussed processes where one membrane splits into multiple membranes, or membranes mix with scattering states made of large bound states of zero branes.

It will be interesting to look for a gravitational dual for the decoupled gauge theory of section 4. One possibility is to start with a time dependent multi-$D0$ brane solution of the type considered in [50, 51]. Then a $D2$-brane could be introduced as a perturbation, as the $D5$-brane was introduced in a background of $D3$ branes in [52]. The gravitational dual may shed light on the strong coupling regime of $R \to 0$. Another approach for a spacetime gravitational description is to consider the spherical $D0$-$D2$ system as a spherical shell which causes a discontinuity in the extrinsic curvature due to its stress tensor, and acts as a monopole source for the two-form field strength due to the $D0$-branes, and a dipole source for the four-form field strength due to the spherical $D2$-brane. As long as the $D0$-brane fluid description is valid, it should be possible to view the $D0$ branes as smeared on a sphere of time-dependent radius. Exploring the solutions and regimes of validity of these different gravitational descriptions will undoubtedly be a very interesting avenue for the future. The recent paper [53] is an example where a gauge theory dual in a time-dependent set up is proposed.

Any discussion of (3.1) will certainly remind many readers of symmetrised trace issues, such as those raised by [54]. The system considered here belongs to a class of configurations which come in families labeled by a size N of matrices where $[\Phi_i, \Phi_j]$ goes to zero in the large-N limit. Higher even dimensional fuzzy spheres and co-adjoint orbits also belong to this class. In these cases the large-N limit often has some sort of abelian geometrical description. For the leading large-N in these cases the ordering of Matrices does not matter and, at the level of classical equations of motion, the system can be compared with an abelian dual [10, 11]. Here we have extended the comparison to fluctuations and found agreement. It will be interesting to see if the comparison can be extended to higher orders in $1/N$, where an appropriate star product is used on the higher brane and the Matrix product on the $D0$-brane side is interpreted as a star product on the sphere. A successful identification will require the correct implementation of the symmetrised trace at higher orders. Given the subtleties of separating field strengths and derivatives in the non-abelian case (discussed for example in [55]), it is probably best to approach the question of symmetrised trace and its corrections by embedding the non-abelian system of interest into a family of systems labeled by a matrix size N which which can be taken to be large and admits an abelian limit.

In this paper we have focused on the analysis of fluctuations in the case of time dependent $D0$-$D2$ solutions. A similar analysis can be performed in the spatial $D1 \perp D3$ configurations. Some aspects of this problem have already been studied in [56]. In the case

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of systems involving higher dimensional fuzzy spheres, such as $D0$-$D4$ ($D1\perp D5$) systems or $D0$-$D6$ ($D1\perp D7$) systems, we expect on general grounds that there will be an abelian description based on a geometry of the form $\mathrm{SO}(2k+1)/\,\mathrm{U}(k)$ and a non-abelian description on the S^{2k} [13, 14, 57, 58]. A detailed fluctuation analysis of the kind studied here should allow a more precise description of strong and weak coupling regimes. The flat space limit of our analysis of fluctuations about fuzzy sphere solutions should be related to the work in [59, 60].

Another interesting avenue is to use the quadratic action we have obtained to do one and higher loop computations of the partition function and correlators. As indicated by the connections to integrability in section 2.2, these computations have interesting mathematical structure. It will also be interesting to incorporate the non-linearities in the fields perturbatively in the region of R close to R_0.

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A. Useful formulas for derivation of fluctuation action from D0-branes

A list of useful formulas is the following

$$\begin{aligned}
[\Phi_i,\Phi_j][\Phi_j,\Phi_i] &= 8N^2\hat{R}^4 + 8\hat{R}^2\,(\partial_a\phi)\,(\partial^a\phi) + 48\hat{R}^2\phi^2 + 32\hat{R}^3N\phi - 48\hat{R}^3\frac{\epsilon^{ab}}{\sin\theta}F_{ab}\,\phi \\
&\quad +32\hat{R}^3\frac{\epsilon^{ab}}{\sin\theta}A_b\,(\partial_a\phi) - 16\hat{R}^4N\frac{\epsilon^{ab}}{\sin\theta}F_{ab} + 16\hat{R}^4F_{ab}F^{ab} + 64\hat{R}^4A_aA^a \\
&\quad -64\hat{R}^4\frac{1}{\sin^2\theta}\left[\epsilon^{ab}(\partial_\theta A_a)(\partial_\phi A_b) + A_\phi\,(\partial_\phi A_\theta)\cot\theta\right] \qquad (A.1)\\
(\partial_t\Phi_i)\,(\partial_t\Phi_i) &= N^2\dot{\hat{R}}^2 - 4\hat{R}^2F_{0a}F^{0a} + \dot{\phi}^2 + 2\dot{\hat{R}}N\dot{\phi} \\
&\quad +4(\dot{\hat{R}})^2A_aA^a + 4\hat{R}\dot{\hat{R}}\,\partial_t\,(A_aA^a) \qquad (A.2)\\
(\partial_t\Phi_i)[\Phi_i,\Phi_j] &= 2i\hat{R}\dot{\hat{R}}NK^a_j(\partial_a\phi) + 4i\hat{R}^3N\epsilon_{ijp}x_pK^a_i(\partial_tA_a) \qquad (A.3)\\
(\partial_t\Phi_i)[\Phi_i,\Phi_j][\Phi_j,\Phi_k](\partial_t\Phi_k) &= \\
&= 4\hat{R}^2\dot{\hat{R}}^2N^2\hat{h}^{ab}(\partial_a\phi)(\partial_b\phi) + 16\hat{R}^6N^2\hat{h}^{ab}(\partial_tA_a)(\partial_tA_b) \\
&\quad +8\dot{\hat{R}}\hat{R}^4N^2\omega^{ab}(\partial_tA_a)(\partial_b\phi) - 8\dot{\hat{R}}\hat{R}^4N^2\omega^{ab}(\partial_a\phi)(\partial_tA_b). \qquad (A.4)
\end{aligned}$$

To get the quadratic fluctuations we take a square root, expand, use the matrix correspondence between the trace and the integral over the sphere (3.17), and also employ the equations of motion. Note that, after taking the trace, the terms in the last line in (A.1) will combine with the linear term $-16\hat{R}^4N\frac{\epsilon^{ab}}{\sin\theta}F_{ab}$ to give

$$-16\hat{R}^4N\epsilon^{ab}\left(F_{ab} + i[A_a,A_b] + \frac{2}{N}(\partial_c\omega^{cd})(A_a\partial_dA_b)\right)\ . \qquad (A.5)$$

We see that F_{ab} gives a total derivative while the last two terms are not individually total derivatives but combine as such. The need for additional terms in the field strength, beyond the commutator $[A_a, A_b]$ (defined in (3.19)) was explained in [20]. The terms in (A.5) can be neglected when we are considering topologically trivial fluctuations.

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